# A note on the third invariant factor of the Laplacian matrix of a graph *

Jian Wang, Yong-Liang Pan†

Department of Mathematics, University of Science and Technology of China

Hefei, Auhui 230026, The People's Republic of China

Abstract

Let $G$ be a simple connected graph with $n \geq 5$ vertices. In this note, we will prove that $s_3(G) \leq n$, and characterize the graphs which satisfy that $s_3(G) = n$, $n-1$, $n-2$, or $n-3$, where $s_3(G)$ is the third invariant factor of the Laplacian matrix of $G$.

**Keywords** Graph; Laplacian matrix; Invariant factor; Smith normal form.

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Let $G = (V, E)$ be a simple connected graph with vertex set $V = V(G) = \{v_1, \cdots, v_n\}$ and edge set $E = E(G)$. Denote the degree of vertex $v_i$ by $d_i$ and let $D(G) = \mathrm{diag}(d_1, \cdots, d_n)$. The Laplacian matrix is $L(G) = D(G) - A(G)$, where $A(G)$ is the $(0,1)-$adjacency matrix of $G$.

Denote by $\Delta_i(G)$ the $i-$th determinantal minors of $L(G)$, i.e., the greatest common divisor of all the $i-$by$-i$ determinantal minors of $L(G)$. Of course $\Delta_i(G) \mid \Delta_{i+1}(G)$, $0 < i < n$. The invariant factors of $L(G)$ are defined by $s_{i+1}(G) = \frac{\Delta_{i+1}(G)}{\Delta_i(G)}$, $0 \leq i < n$, where $\Delta_0(G) = 1$. It is easy to see that $s_i(G) | s_{i+1}(G)$, $1 \leq i \leq n-1$, and $s_n(G) = 0$ since $L(G)$ is singular. The Smith normal form of $L(G)$ is the $n-$square diagonal matrix $F(G)$ whose $(i,i)$ entry is $s_i(G)$. It follows from the well known matrix-tree theorem that $\Delta_{n-1}(G) = s_1(G)s_2(G)\cdots s_{n-1}(G)$ is equal to the spanning tree number of $G$. So the invariant factors of $G$ can be used to distinguish pairs of non-isomorphic graphs which have the same spanning tree number, and so there is considerable interest in their properties.

Since $G$ is a simple graph, its invariant factor $s_1(G)$ must be equal to 1, however most of the others are not easy to be determined. From the following lemma, we

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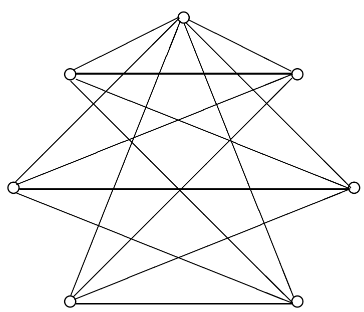
†Corresponding author. Email: ylpan@ustc.edu.cn

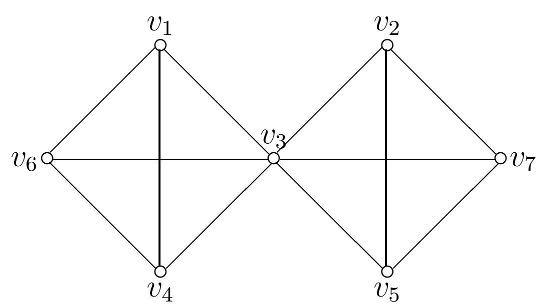
know that $s_2(G) = 1$ if $G$ is not the complete graph $K_n$, while $F(K_n) =$diag$(1, n, \cdots, n, 0)$. In this note, we will show that $s_3(G) \leq n$, and characterize the graphs which satisfy that $s_3(G) = n,\ n-1,\ n-2,$ or $n-3$.

**Lemma** ([1]) For a simple connected graph $G$ with order $n \geq 3$, $s_2(G) \neq 1$ if and only if $G$ is the complete graph $K_n$, which has $s_i(G) = n$, $2 \leq i \leq n-1$.

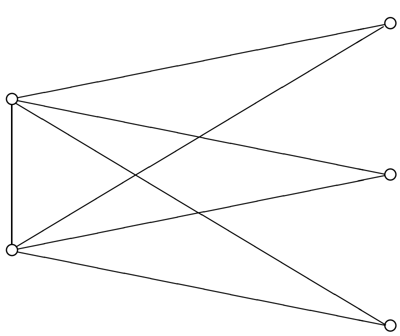
In the following theorem, $v \cdot G$ denotes the graph obtained by adding an edge joining some vertex of $G$ to a further vertex $v$; $G - 2e$ denotes the graph obtained from $G$ by deleting two edges which have no common vertex; $G - C_4$ denotes the graph obtained from $G$ by deleting a circle of length 4; $G - 2C_3$ denotes the graph obtained from $G$ by deleting 6 edges in two cycles of length 3 which have no common vertices (See Fig. 1). In the proof of the following theorem, $x \sim y$ means that the vertices $x$ and $y$ are adjacent and $x \not\sim y$ means that they are not adjacent.



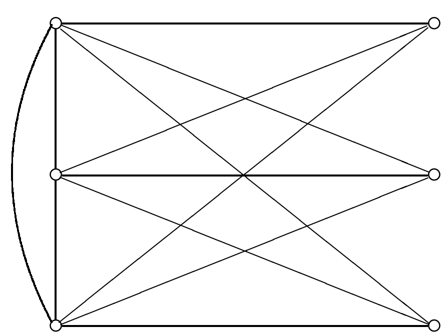
$K_7 - 2C_3$



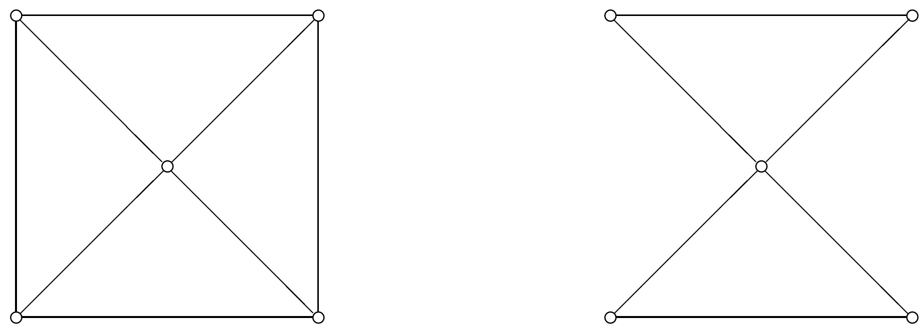

$K_7 - K_{3,3}$



$K_5 - C_3$



$K_6 - C_3$



$K_5 - 2e$ $K_5 - C_4$

Fig. 1

**Theorem** Let $G \neq K_n$ be a simple connected graph with order $n \geq 5$. Then $s_3(G) \leq n$. Moreover, $s_3(G) = n$ if and only if $G = K_n - e$, where $e$ is an edge of $K_n$; $s_3(G) = n-1$ if and only if $G = v \cdot K_{n-1}$; $s_3(G) = n-2$ if and only if $n = 5$ and $G = K_5 - 2e$ or $G = K_5 - C_4$; $s_3(G) = n-3$ if and only if $G$ is one of the following 6 graphs: $K_{2,3}$, $K_5 - C_3$, $K_6 - C_3$, $K_7 - 2C_3$, $K_{3,3}$ and $K_7 - K_{3,3}$.

**Proof** Since $G \neq K_n$, then its diameter is at least 2. In fact, we only need to consider the graphs with diameter 2, since if the diameter of $G$ is more than 2 then by theorem 4.5 in [1] we have that $s_3(G) = 1$. Let $v_1$ and $v_2$ be two nonadjacent vertices in $G$. There is a further vertex $v_3$ which is adjacent to both $v_1$ and $v_2$. Now we need to distinguish 8 cases to go on the argument.

*Case* 1. Some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ satisfies that $v_4 \not\sim v_1$, $v_4 \not\sim v_2$, $v_4 \not\sim v_3$. Since $G$ is connected, there is some vertex $v_5$ in $V(G)/\{v_1, v_2, v_3, v_4\}$ adjacent to both $v_1$ and $v_4$. Clearly, $\det(L[1,3,4|2,3,5]) = -1$, where $L[1,3,4|2,3,5]$ is the submatrix of $L(G)$ that lies in the rows corresponding to vertices $v_1, v_3, v_4$ and columns corresponding to vertices $v_2, v_3, v_5$. Therefore $s_3 = 1$.

*Case* 2. Some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ satisfies that $v_4 \not\sim v_1$, $v_4 \not\sim v_2$, $v_4 \sim v_3$. In this case, $\det(L[1,2,3|2,3,4]) = d_2 \leq n-2$. So $s_3 \leq n-2$.

*Case* 3. Some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ satisfies that $v_4 \not\sim v_1$, $v_4 \sim v_2$, $v_4 \not\sim v_3$. In this case, $\det(L[1,2,3|2,3,4]) = 1$, and hence $s_3 = 1$.

*Case* 4. Some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ Satisfies that $v_4 \not\sim v_1$, $v_4 \sim v_2$, $v_4 \sim v_3$. In this case, $|\det(L[1,2,3|2,3,4])| = d_2 + 1 \leq n-1$. Hence $s_3 \leq n-1$.

*Case* 5. Some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ satisfies that $v_4 \sim v_1$, $v_4 \not\sim v_2$, $v_4 \not\sim v_3$. In this case, very similar to *case* 3, we have $s_3 = 1$.

*Case* 6. Some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ satisfies that $v_4 \sim v_1$, $v_4 \not\sim v_2$, $v_4 \sim v_3$. In this case, very similar to *case* 4, we have $s_3 \leq n-1$.

*Case* $1-$*Case* 6 show that if some vertex $v_4$ in $V(G)/\{v_1, v_2, v_3\}$ is not adjacent to both $v_1$ and $v_2$, then $s_3 < n$. So, we will only need to deal with the cases in which every further vertex in $V(G)/\{v_1, v_2, v_3\}$ is adjacent to both $v_1$ and $v_2$.

*Case* 7. Every vertex in $V(G)/\{v_1, v_2, v_3\}$ is adjacent to both $v_1$ and $v_2$, and at least one vertex $v_4$ is not adjacent to $v_3$. In this case, we distinguish 3 subcases.

*Subcase* 1. There is some vertex $v_5$ in $V(G)/\{v_1, v_2, v_3, v_4\}$ adjacent to all of the vertices $v_1, v_2$ and $v_3$. Then $\det(L[1,2,3|1,4,5]) = d_1 = n-2$. Hence $s_3 \leq n-2$.

*Subcase* 2. Every vertex in $V(G)/\{v_1, v_2, v_3\}$ is not adjacent to $v_3$, and the induced subgraph $G[v_4, \cdots, v_n] \neq K_{n-3}$. If we choose any two nonadjacent vertices in $\{v_4, \cdots, v_n\}$ as $v_4$ and $v_5$, then we have $-\det(L[1,4,5|1,3,5]) = -\begin{vmatrix} d_1 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & d_5 \end{vmatrix} = d_5$. Hence we have that $s_3 \leq d_5 \leq n-3$.

*Subcase* 3. Every vertex in $V(G)/\{v_1, v_2, v_3\}$ is not adjacent to $v_3$, but $G[v_4, \cdots, v_n] = K_{n-3}$. It is not difficult to obtain that $F(G) =$diag$(1,1,1,n-1,\cdots,n-1,\ 2(n-1)(n-2),0)$.

*Case* 8. Every vertex $G-\{v_1, v_2, v_3\}$ is adjacent to all of the vertices $v_1$, $v_2$ and $v_3$. In this case, we distinguish two subcases.

*Subcase* 1. $G-\{v_1, v_2, v_3\} \neq K_{n-3}$, then there are two nonadjacent vertices $v_4$ and $v_5$ in $V(G)/\{v_1, v_2, v_3\}$. It follows that $\det(L[2,3,4|1,4,5]) = d_4 \leq n-2$. Hence $s_3 \leq n-2$. (In fact, if we regard the vertices $v_4$ as $v_1$, $v_5$ as $v_2$, $v_1$ as $v_3$, $v_2$ as $v_4$, and $v_3$ as $v_5$, then we are back in the subcase 1 of case 7.)

*Subcase* 2. $G[v_4, \cdots, v_n] = K_{n-3}$. Then $G = K_n - e$. It is not difficult to obtain $F(K_n - e)$=diag$(1,1,n,\cdots,n,\ n(n-2),0)$.

From above argument we have that $s_3 \leq n$ and $s_3 = n$ if and only if $G$ is $K_n - e$.

Clearly, case 6 is symmetric to case 4, the required graphs in case 4 are the isomorphic to the required graphs in case 4.

From proposition 1 in [2], we know that $F(v \cdot K_{n-1}) =$diag$(1,1,n-1,\cdots,n-1,0)$. Now we prove the converse: if $s_3(G) = n-1$ then $G = v \cdot K_{n-1}$.

From the argument of the above 8 cases, it follows that if $s_3(G) = n-1$ then every vertex in $V(G)/\{v_1, v_2, v_3\}$ is adjacent to $v_3$ and only *case* 4 or *case* 6 may occur. If *case* 4 occurs, then $|\det L[1,2,3|2,3,4]| = d_2 + 1 \leq n-1$. It follows from $s_3(G) = n-1$ that $d_2 = n-2$ and then *case* 6 will never occur. Similarly, if *case*

6 occurs then *case* 4 will never occur. Without loss of generality, we assume that only *case* 4 occurs. We need to deal with two subcases here.

*Subcase* 1. There are two vertices in $\{v_4, \cdots, v_n\}$ which are not adjacent. We regard the two nonadjacent vertices as $v_1$, $v_2$, and regard $v_1$ as $v_4$, we are then back in *case* 2, so we have that $s_3 \leq n-2$.

*Subcase* 2. $G[v_4, \cdots, v_n] = K_{n-3}$. Note that $v_2$ and $v_3$ are adjacent to every vertex in $V(G)/\{v_1, v_2, v_3\}$, thus $G = v \cdot K_{n-1}$.

A direct calculation can show that $F(K_5 - 2e)$ =diag $(1, 1, 3, 15, 0)$ and $F(K_5 - C_4)$ =diag$(1, 1, 3, 3, 0)$. Now we prove that if $s_3 = n-2$, then $n = 5$ and $G = K_5 - 2e$ or $K_5 - C_4$.

By the above argument, we know that if $s_3 = n-2$ then *case*s 1, 3 and 5 may not occur.

If *case* 2 occurs, then $\det(L[1, 2, 3|2, 3, 4]) = d_2 \leq n-2$. So $d_2$ must be $n-2$. It is a contradiction to *case* 2.

If *case* 4 occurs, then $\det(L[1, 2, 3|2, 3, 4]) = d_2 + 1 \leq n-1$, and hence $d_2 = n-3$. Then we must have exact one vertex $v_5$ in *case* 6. If $n > 5$, there is another vertex $v_6$ in *case* 7 or *case* 8. We have, if $v_6 \not\sim v_3$, then $\det(L[1, 2, 3|4, 5, 6]) = 2 < n-2$; and if $v_6 \sim v_3$, then $\det(L[1, 2, 3|4, 5, 6]) = 1 < n-2$. So $n = 5$. Now, $v_5 \sim v_1$, $v_5 \not\sim v_2$, $v_5 \sim v_3$. If $v_4 \not\sim v_5$, then $G = K_5 - C_4$, whose Smith normal form is diag$\{1, 1, 3, 3, 0\}$; if $v_4 \sim v_5$, then $G = K_5 - P_4$, whose Smith normal form is diag$\{1, 1, 1, 21, 0\}$. Impossible.

If *case* 7 occurs, then from the above argument, we know that only its *subcase* 1 occurs. Note that $\det(L[1, 2, 3|1, 4, 5]) = \begin{vmatrix} d_1 & -1 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{vmatrix} = d_1 \leq n-2$ and $\det(L[1, 2, 3|2, 4, 5]) = \begin{vmatrix} 0 & -1 & -1 \\ d_2 & -1 & -1 \\ -1 & 0 & -1 \end{vmatrix} = -d_2$, so $d_1 = d_2 = n-2$. Moreover, $\det(L[2, 3, 4|1, 3, 5]) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & d_3 & -1 \\ -1 & 0 & x \end{vmatrix} = -(d_3 + 1 + x)$ and $\det(L[2, 3, 4|1, 4, 5]) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & d_4 & x \end{vmatrix} = d_4 - 1 - x \leq n-2$, where $x$ is 0 if $v_4 \not\sim v_5$, or $-1$ if $v_4 \sim v_5$. If $x = 0$, then $d_4 - 1 = n-2$ and it follows that $d_4 = n-1$, impossible. So $x = -1$, and then $d_3 = d_4 = n-2$. For $i \geq 5$, $\det(L[1, 4, i|2, 3, i]) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & d_i \end{vmatrix} = -(d_i + 2)$. So $n-2 \leq d_5 + 2 \leq n+1$.

- If $d_5 + 2 = n+1$, then $n-2$ divides $n+1$, thus $n = 5$ and hence $G = K_5 - 2e$.

- If $d_5 + 2 = n$, then $n-2$ divides $n$. Thus $n = 4$, a contradiction.
- If $d_5+2 = n-2$, then there are further 3 vertices $v_6$, $v_7$ and $v_8$ not adjacent to $v_5$. Now $\det(L[3,5,6|1,4,5]) = \begin{vmatrix} -1 & 0 & -1 \\ -1 & -1 & d_5 \\ -1 & -1 & 0 \end{vmatrix} = -d_5 = n-4$. Now $n-4$ divides $n-2$, it follows that $n = 5$, or 6. Impossible.

If *case* 8 occurs, then its *subcase* 1 occurs and *subcase* 2 does not. Then we only need to deal with *subcase* 1 of *case* 7, it has been done.

With the aid of Maple, we obtain the Smith normal forms of the graphs $K_{2,3}$, $K_5{-}C_3$, $K_6{-}C_3$, $K_7{-}2C_3$, $K_{3,3}$ and $K_7{-}K_{3,3}$ as follows: $F(K_{2,3}) = \mathrm{diag}(1,1,2,6,0)$, $F(K_5{-}C_3) = \mathrm{diag}(1,1,2,10,0)$, $F(K_6{-}C_3) = \mathrm{diag}(1,1,3,6,18,0)$, $F(K_7{-}2C_3) = \mathrm{diag}(1,1,4,4,4,28,0)$, $F(K_{3,3}) = \mathrm{diag}(1,1,3,3,9,0)$, $F(K_7{-}K_{3,3}) = \mathrm{diag}(1,\ 1,\ 4,\ 4,\ 4,\ 4,\ 0)$. In the following, we will prove that if $s_3 = n-3$ then $G$ must be one of these 6 graphs.

By the above argument, we know that *cases* 1, 3, 5 can not occur.

If *case* 2 occurs, then $\det(L[1,2,3|2,3,4]) = d_2$, $\det(L[1,2,3|1,3,4]) = d_1$ and $\det(L[2,3,4|1,3,4]) = -d_4$. Hence $d_1 = d_2 = d_4 = n-3$. Consider the number of vertices with degree $n-1$, we distinguish 3 subcases.

*Subcase* 1. $G$ has at least 3 vertices with degree $n-1$, then $L(G)$ has a submatrix $L_1 = \begin{pmatrix} (n-3)I_3 & -J_3 \\ -J_3 & nI_3 - J_3 \end{pmatrix}$, where $I_3$ is the $3\times 3$ identity matrix, $J_3$ is the $3 \times 3$ all 1's matrix. Note that $\det(L_1[1,4,6|2,4,5]) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & n-1 & -1 \\ -1 & -1 & -1 \end{vmatrix} = -n$. So $n-3$ divides $n$, it follows that $n = 6$ and hence $G = K_6 - C_3$.

*Subcase* 2. $G$ has 1, or 2 vertices with degree $n-1$.

- If $n = 5$, then clearly, $G = K_5 - C_3$.
- If $n \geq 6$, then suppose $v_i \not\sim v_j$, where $v_i$, $v_j \in V(G)/\{v_1, v_2, v_4\}$. $L(G)$ has a submatrix $L_2 = \begin{pmatrix} (n-3)I_3 & -J_3 \\ -J_3 & B \end{pmatrix}$, where $B = \begin{pmatrix} d_i & 0 & -1 \\ 0 & d_j & -1 \\ -1 & -1 & n-1 \end{pmatrix}$. Then $|\det(L_2[1,4,6|2,4,5])| = |\begin{vmatrix} 0 & -1 & -1 \\ -1 & d_i & 0 \\ -1 & -1 & -1 \end{vmatrix}| = d_i \leq n-2$. So $d_i = n-3$. In the same way, we can get $d_j = n-3$. Thus the vertices of $G$ share two degrees: $n-1$ or $n-3$. $\det(L_2[2,4,6|3,5,6]) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & n-1 \end{vmatrix} = -(n+1)$. Hence $n-3$ divides $n+1$, then $n = 7$ and it follows that $G = K_7 - 2C_3$.

*Subcase* 3. $G$ has no vertex with degree $n-1$.

- If $n=5$, clearly $G=K_5-C_3-e=K_{2,3}$.
- If $n=6$, clearly $G=K_6-2C_3=K_{3,3}$.
- If $n\geq 7$, then $L(G)$ has a principal submatrix $L_3=\begin{pmatrix}(n-3)I_3 & -J_{3\times 4}\\ -J_{4\times 3} & C\end{pmatrix}$,

where $C=\begin{pmatrix} d_i & 0 & y_1 & y_3\\ 0 & d_j & y_2 & y_4\\ y_1 & y_2 & d_u & y_5\\ y_3 & y_4 & y_5 & d_v\end{pmatrix}$ with $y_i=0$ or $-1$. Note that $\det(L_3[1,4,7|3,5,6])=$ $y_1+y_4-y_5$, $\det(L_3[2,4,6|3,5,7])=y_2+y_3-y_5$. Now $(n-3)\mid(y_1+y_4-y_5)$ and $(n-3)\mid(y_2+y_3-y_5)$, it follows that $y_1=y_2=y_3=y_4=y_5=0$ and hence $d_i\leq n-4$. Now $\det(L_4[1,4,7|3,4,6])=\begin{vmatrix}0 & -1 & -1\\ -1 & d_i & 0\\ -1 & 0 & 0\end{vmatrix}=-d_i$. Therefore $n-3$ divides $d_i$. But $d_i\leq n-4$, so it is impossible.

If *case* 4 occurs, then $|\det(L[1,2,3|2,3,4])| = d_2+1$. Therefore we have $n-3\leq d_2+1\leq n-1$. If $d_2+1=n-1$, then $n-3$ divides $n-1$, thus $n=5$ and $d_2=n-2=3$. So $v_5$ must be in case 4. If $v_4\sim v_5$, then $G=v\cdot K_4$, whose $F(G)$=diag$(1,1,4,4,0)$. It is impossible. If $v_4\not\sim v_5$, then a direct calculation can show $F(G)$=diag$(1,1,1,8,0)$, it is a contradiction. So $d_2+1=n-3$. There must be some vertex $v_5$ in *case* 6 and hence we have $-\det(L[1,2,3|1,3,5])=$ $-\begin{vmatrix}d_1 & -1 & -1\\ 0 & -1 & 0\\ -1 & d_3 & -1\end{vmatrix}=d_1+1\leq n-1$, so $d_1=n-4$. Then there are two vertices $v_6$ and $v_7$ such that $v_7$ together with $v_5$ are in *case* 6, and $v_6$ together with $v_4$ are in *case* 4. Then $-\det(L[1,2,3|3,6,7])=-\begin{vmatrix}-1 & 0 & -1\\ -1 & -1 & 0\\ d_3 & -1 & -1\end{vmatrix}=d_3+2$. Thus we have that $n-3$ divides $d_3+2$ and $n-3\leq d_3+2\leq n+1$.

- If $d_3+2=n+1$, then $n=7$. Note that $\det(L[1,2,4|1,3,5])=\begin{vmatrix}n-4 & -1 & -1\\ 0 & -1 & 0\\ 0 & -1 & x\end{vmatrix}=$

$-x(n-4)$, where $x=-1$, or $0$. Since $(n-3)\mid -x(n-4)$ then $x=0$. So $v_4\not\sim v_5$. In the same way, we can see that $v_4\not\sim v_7$, $v_5\not\sim v_6$ and $v_7\not\sim v_6$. So there is no edges between the vertices $v_1,v_5,v_7$ and $v_2,v_4,v_6$. Moreover, we have $d_5\leq n-4$. Note that $\det(L[2,3,5|3,5,7])=\begin{vmatrix}-1 & 0 & 0\\ -1 & -1 & -1\\ -1 & d_5 & y\end{vmatrix}=y-d_5$, where $y=-1$, or $0$. From $(n-3)\mid(y-d_5)$, we can get $d_5=n-4$ and $y=-1$. So $v_5\sim v_7$. In the same way, we can get $v_4\sim v_6$. Thus $G=K_7-K_{3,3}$ (See Fig.1).

- If $d_3+2=n$ or $n-1$, then $n=6$ or $5$ respectively, impossible.
- If $d_3+2=n-3$, then $d_3=n-5$. Then there exists a vertex $v_8$ such that

$v_1 \sim v_8$, $v_2 \sim v_8$ and $v_3 \not\sim v_8$. Thus we have $\det(L[1,2,3|6,7,8]) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} = -2$. From $(n-3)|2$ we get $n = 5$. Impossible.

Now we assume that *case* 7 occurs. We know only its subcase 1 and subcase2 may occur. If its *subcase* 1 occurs, then $\det(L[1,2,3|1,4,5]) = \begin{vmatrix} d_1 & -1 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{vmatrix} = d_1 = n-2$. So $n-3$ divides $n-2$, impossible. If its *subcase* 2 occurs, if we regard the vertices $v_4$ as $v_1$, $v_5$ as $v_2$, $v_1$ as $v_3$, $v_3$ as $v_4$, then we are back in case 2. The required graphs have been determined.

If *case* 8 occurs, then only its subcase 1 may occur. Of course, we are back in the subcase 1 of case 7 and the required graphs have been determined.

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